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THE STABILITY OF A PLANE IDEAL RIGID–PLASTIC COUETTE FLOW†

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The stability of a plane Couette flow in an ideal rigid–plastic layer with respect to small perturbations is considered. Sufficient integral estimates of the stability are given for an arbitrary unperturbed velocity profile. The spectral boundary-value problem is solved analytically in the domain of the most unstable long wavelengths.

Shear flows have been investigated for various rheological models of bodies (for a non-Newtonian second order model in [1–3], where the constitutive relations are chosen in the Rivlin–Ericksen form [4], for viscoelastic liquids in the short memory approximation in [5], for ideal plastic anisotropic media in [6] and for viscoplastic media in [7]). The class of perturbations, with respect to which the stability of a system is studied, plays a substantial role. For instance, it has been shown in [1] that the non-Newtonian properties, elasticity and non-linearity, destabilize a flow. It has been established using energy methods [2] that, in the case of two-dimensional perturbations, elastoviscosity always stabilizes a flow while, in the case of spatial perturbations, its role is determined by a dimensionless relation between the parameters of the model. We also mention the monograph [8] in which these questions have been covered in detail.

1. FORMULATION OF THE SPECTAL BOUNDARY-VALUE PROBLEM

We shall make use of the results in [9] where an equation was derived for the one-dimensional shear stability in an incompressible viscoplastic layer with respect to two-dimensional perturbations (a generalized Orr–Sommerfeld equation). We will write this for an ideal rigid–plastic, Saint Venant medium for which the relation between the stress tensor σ_{mj} and the strain rate tensor υ_{mj} is specified by the relationships

$$\sigma_{mj} = -p\delta_{mj} + 2\tau\upsilon_{mj}/U, \quad m,j=1,3 \tag{1.1}$$

where p is a function of the pressure, U is the maximum shear rate, τ is the shear yield stress [7] and δ_{mj} is the unit tensor. The stability equation has the form

$$4\tau s(\varphi'/U)' + (\alpha/s + i\upsilon)(\varphi'' - s^2\varphi) - i\upsilon''\varphi = 0 \tag{1.2}$$

Here φ is the complex amplitude of the stream function ψ: $\psi(x_1, x_3, t) = \varphi(x_3)\exp(isx_1 + \alpha t)$, $s \in \mathbf{R}$, $\alpha_* + i\alpha_{**} = \alpha \in \mathbb{C}$, $\upsilon(x_3)$ is the velocity of the principal motion, specified in the whole layer $\{\Omega: 0 < x_3 < 1\}$ and $U \equiv |\upsilon'|$. All the quantities in (1.1) and (1.2) are reduced to dimensionless form in the basis

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$\{\rho, V, h\}$, here ρ is the constant density of the body, V and h are the characteristic velocity and the linear dimensions and $\tau = \tau_s/(\rho V^2)$ (τ_s is the dimensional shear yield stress). Derivatives with respect to x_3 are denoted by primes.

When $\tau = 0$, Eq. (1.2) reduces to the Rayleigh equation, which has been investigated in detail. Theorems from the theory of hydrodynamic stability of an ideal fluid are applicable to the corresponding shear flow. These are the Rayleigh and Fjortjoft–Hoyland conditions and Howarth's theorem "on a semicircle" [10], which are associated with the sign of the curvature of the velocity profile within a domain of a flow.

We know that non-uniqueness of the solution of the equations of motion of viscoplastic and ideally plastic bodies is possible. Generally speaking, there is an infinitely large number of values of U which correspond to one and the same shear stress τ. Consequently, in the case of flow between plane, parallel boundaries moving along the x_1 axis with constant velocities (plane Couette flow), any monotonic function $\upsilon(x_3)$, which takes specified values at the points 0 and 1, may be selected as the main profile. Any distribution of rigid zones ($U = 0$) in the interval $0 < x_3 < 1$ is also possible. In this case, the boundary conditions in the perturbations have the form

$$x_3 = 0:\ \varphi = 0;\quad x_3 = 1:\ \varphi = 0 \tag{1.3}$$

We will next assume that the principal flow is characterized by an arbitrary, monotonically increasing function $\upsilon(x_3)$ with a continuous derivative such that $U \leq q$ and

$$\int_0^c \frac{dx_3}{U(x_3)} < \infty, \quad \forall c \in [0,1]$$

if the tangential velocity undergoes discontinuities (zones of plastic flow with a monotonic and smooth profile alternate with rigid interstratifications, where the strain rate is zero), it is then sufficient to investigate the stability in each of these flow zones independently. In order to do this, the integration limits [0, 1] in the subsequent operations must be replaced by other limits which are determined experimentally, for example, from the main motion. Here, the degenerate case when there are no such flow zones is also possible when the strain rate is the sum of certain δ-functions. This case is precluded from the present treatment.

2. INTEGRAL ESTIMATES OF STABILITY IN $\overline{H}_2(\Omega)$ SPACE

Let φ be an element of a complex-valued Hilbert space $\overline{H}_2(\Omega)$ with the norm

$$\|\varphi\| = \left(\int |\varphi''|^2\, dx_3\right)^{1/2} \tag{2.1}$$

which has two continuous derivatives [11] (here and everywhere henceforth, integration with respect to x_3 is carried out from 0 to 1).

Let us multiply (1.2) by the complex conjugate function $\overline{\varphi}$ and integrate with respect to x_3 from 0 to 1. When the boundary conditions (1.3) are taken into account, we obtain

$$4\tau s I_\upsilon^2 + \alpha(I_1^2 + s^2 I_0^2)/s + i\int[(\upsilon'' + s^2\upsilon)|\varphi|^2 + \upsilon|\varphi'|^2]dx_3 + i\int \upsilon'\varphi'\overline{\varphi}dx_3 = 0 \tag{2.2}$$

$$\left(I_m^2 = \int|\varphi^{(m)}|^2\, dx_3,\quad m = 0,1;\quad I_\upsilon^2 = \int|\varphi'|^2|\upsilon'|^{-1}\, dx_3\right)$$

We now equate the real part of the expression on the left-hand side of (2.2) to zero

$$4\tau s I_\upsilon^2 + \alpha_*(I_1^2 + s^2 I_0^2)s^{-1} - \int \upsilon'(\varphi'\overline{\varphi})_{**}\, dx_3 = 0 \tag{2.3}$$

and make use of the Schwartz inequality in the space $\overline{H}_2(\Omega)$ with the norm (2.1)

$$\int|\upsilon'|\,|\varphi'|\,|\varphi| dx_3 \leq q I_0 I_1 \tag{2.4}$$

The following theorem follows from (2.3) and (2.4).

Theorem. Let $\alpha(s, \tau)$ be an arbitrary characteristic number of the spectral boundary-value problem (1.2), (1.3). Then

$$\alpha_* \leq s(qI_0I_1 - 4\tau sI_\upsilon^2)/(I_1^2 + s^2 I_0^2) \tag{2.5}$$

A negative value of the right-hand side of inequality (2.5) is a sufficient condition for the stability of a plane, ideal plastic Couette flow.

Corollary 1. If $\tau/q^2 > (\pi^2 + s^2)/(8\pi^2 s^2)$, then $\alpha_* < 0$.

Corollary 2. If $\tau/q^2 > 1(4\pi s)$, then $\alpha_* < 0$.

The proof of Corollaries 1 and 2 follows from the obvious inequality $sI_0I_1 \leq (I_0^2 + s^2 I_1^2)/2$ and Friedrich's inequalities in the space $\overline{H}_2(\Omega)$ with the norm (2.1) [11]

$$I_1^2 \geq \pi^2 I_0^2, \quad I_\upsilon^2 \geq I_1^2/q$$

The corresponding critical curves 1 and 2 in the $(s, \tau/q^2)$ plane are shown in Fig. 1. Since Corollaries 1 and 2 are independent and sufficient, the following corollary holds.

Corollary 3. If

$$\frac{\tau}{q^2} > \min\left\{\frac{\pi^2 + s_0^2}{8\pi^2 s_0}; \ \frac{1}{4\pi s_0}\right\} = \frac{1}{4\pi s_0}, \quad \forall s_0 > 0 \tag{2.6}$$

for fixed s_0, then $\alpha_*(s_0, \tau) < 0$.

This means that, if, for a certain s_0, the value of τ/q^2 lies above curve 2 in Fig. 1, then the initial flow is stable with respect to a perturbation with wave number s_0. Since, in a real perturbation $\psi(x_1, x_3, t)$ all the harmonics $s > 0$ are present, it is not possible to give stability estimates which are general, sufficient and independent of s. It can be seen from Fig. 1 that long-wavelength variations will be the most unstable and that an increase in τ/q^2 turns out to have a stabilizing influence.

The stability of short waves ("shallow ripples") and the character of long waves which increases with time has previously been pointed out on a number of occasions and, in particular, in problems associated with the deformation of strata in the Earth's core (in a system of the type of an elastic lithosphere and a viscous or ideal asthenosphere [12] and others).

We will estimate the minimum unstable wavelength λ^* $(\lambda = 2\pi/s)$ in a layer with parameters which are characteristic of geophysical processes $\rho = 3000$ kg/m^3, $V = 10$ cm/year (high rate processes), $q = 10$ and $\tau_s = 10^7$ Pa (at temperatures below 200°C) [13]. According to (2.6), $\lambda^* = 8\pi^2\tau/q^2$, that is, $\lambda^* \approx 3\times10^{20}$. Even for the thinnest layers, such wavelengths exceed any linear dimensions of the Earth. Consequently, in the approximation of an ideal rigid–plastic model of the Couette flow of geomaterials, they are stable for any real perturbation s, and the constraint (2.6) does not have any substantial effect on them.

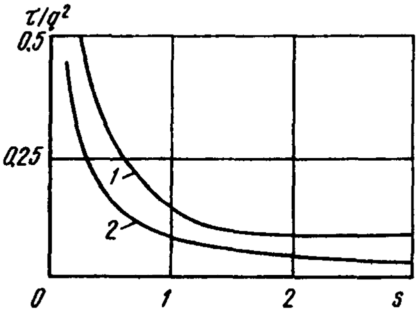

Fig. 1.

3. ANALYTIC SOLUTION OF THE SPECTRAL PROBLEM IN THE LONG-WAVELENGTH APPROXIMATION

Let us now consider the spectral boundary-value problem (1.2), (1.3) in greater detail in the case when $v(x_3)=x_3$ (Couette flow in a narrow sense). Equation (1.2) can then be rewritten in the form

$$(\alpha+isx_3+4\tau s^2)\varphi''-s^2(\alpha+isx_3)\varphi=0 \tag{3.1}$$

Making the following changes in the dependent and independent variables

$$\varphi(x_3)=\eta(\zeta)\exp(\pm sx_3),\quad \zeta=-2(sx_3-i\alpha-4\tau s^2) \tag{3.2}$$

we reduce (3.2) to one of two differential equations with linear coefficients (Laplace equations)

$$\zeta\eta''\mp\zeta\eta'+2i\tau s^2=0 \tag{3.3}$$

the solution of which can be represented in series [14]

$$\eta(\zeta)=\mp\zeta\left(1+\sum_{n=1}^{\infty}\frac{\Xi_n(\pm 2i\tau s^2)}{n!(n+1)!}(\mp\zeta)^n\right)\exp(\pm\zeta) \tag{3.4}$$

$$\Xi_n(z)=(1-z)(2-z)\ldots(n-z)$$

Hence, two fundamental solutions of Eq. (3.1) can be obtained by making the inverse changes of variables and taking the upper and lower signs in formulae (3.4). We substitute the fundamental solutions into the boundary conditions (1.3) and arrive, in a standard way, at the dispersion-wave equation

$$|W_{mj}|=0,\quad m,j=1,2 \tag{3.5}$$

The four elements of the characteristic determinant $|W_{mj}|$ are written in the following form

$$W_{11;21}=\pm\beta-1+\Sigma(s,\beta)$$
$$W_{12;22}=[\pm(\beta-s)-1+\Sigma(s,\beta-s)]\exp(\pm s) \tag{3.6}$$

$$\Sigma(s,\beta)=\sum_{n=1}^{\infty}\frac{\Xi_n(\pm 2i\tau s^2)}{n!(n+1)!}(\pm 2)^n(\pm\beta-1+n)\beta^n,\quad \beta=i(\alpha+4\tau s^2)$$

Since $\alpha_*=\beta_{**}-4\tau s^2$, the stability criterion will have the form

$$\beta_{**}<4\tau s^2 \tag{3.7}$$

We expand the left-hand side of (3.5) in series in s and retain the first three non-zero terms of the expansion

$$|W_{mj}|=s(W_0+W_1 s+(W_2+2i\tau W_2^{(\tau)})s^2+\ldots) \tag{3.8}$$

The coefficients W_0, W_1, W_2, $W_2^{(\tau)}$, which depend solely on β, are products of double series. However, they can be successfully summed in finite form [15]. Omitting the algebra when $\beta\neq 0$ we have

$$W_0=-(\mathrm{ch}\,2\beta-1-\beta^2)/\beta^2,\quad W_1=(1+\beta\,\mathrm{sh}\,2\beta-\mathrm{ch}\,2\beta)/\beta^3$$

$$W_2=-[3(2+\beta)(\mathrm{ch}\,2\beta-1)-6\beta\,\mathrm{sh}\,2\beta-\beta^4]/(6\beta^4)$$

$$W_2^{(\tau)}=\{[2(C+\ln 2\beta)-\mathrm{Ei}\,2\beta-\mathrm{Ei}(-2\beta)]\mathrm{sh}\,2\beta+[\mathrm{Ei}\,2\beta-\mathrm{Ei}(-2\beta)](\mathrm{ch}\,2\beta-1-\beta^2)\}/\beta^2+(\mathrm{ch}\,2\beta-1)/\beta$$

and, when $\beta\neq 0$, $W_0=-1$, $W_1=W_2^{(\tau)}=0$, $W_2=-1/6$. Here, C is Euler's constant and $\mathrm{Ei}(z)$ is the exponential integral.

We also represent $\beta(s,\tau)$ in the form of a series in s

$$\beta(s,\tau)=\sum_{n=0}^{\infty}\beta_n(\tau)s^n \tag{3.9}$$

In order to find $\beta_n(\tau)$, it is necessary to substitute (3.9) into (3.8) and then (3.8) into (3.5) and equate the coefficients of powers of s to zero. On carrying out this operation three times, we obtain

$$\beta_0=\pm i\kappa_0,\ \ \beta_1=\tfrac{1}{2},\ \ \beta_2=-iA_1/A_2,\ \ \kappa_0\in\mathbb{R} \tag{3.10}$$

$$A_1=\pm\left(\frac{1}{8}-\frac{1}{12}\kappa_0^2\right)+\tau\kappa_0^3+\tau\sum_{n=1}^{\infty}\frac{(-1)^n(2\kappa_0)^{2n}}{n(2n)!}\sin 2\kappa_0$$

$$A_2=\sin 2\kappa_0-\kappa_0$$

where $\kappa_0=1.39155$ is the positive root of the equation $\cos\,2\kappa=1-\kappa^2$. When account is taken of (3.10), the condition of stability is written in the following form

$$\pm\kappa_0-A_1s^2/A_2<4\tau s^2$$

or approximately

$$\tau>\pm(0.5895/s^2-0.015)$$

Since we have confined ourselves to just three terms of the asymptotic expansions (3.8) and (3.9), the result obtained holds if $s^3\ll 1$ (the long-wave approximation).

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